Supplementary appendix (not for publication): "A note on expectational stability under non-zero trend inflation"

A The full NKPC under non-zero trend inflation

In Cogley and Sbordone (2008, AER), the full version of the NKPC is given as

$$\pi_t = \zeta mc_t + \tilde{b}_1 E_t \pi_{t+1} + \tilde{b}_2 E_t \sum_{j=2} \phi_1^j \pi_{t+j} + b_3 E_t \sum_{j=0} \phi_1^j (q_{t+j,t+j+1} + \tilde{y}_{t+j+1} - \tilde{y}_{t+j}), \quad \text{(A.1)}$$

where $\tilde{y}_t$ and $q_{t+j,t+j+1}$ denote the (logged) level of output and the real stochastic discount factor, respectively.[1] The degree of price indexation is set at 0. Here, suppose that the utility function is given by the CRRA form: $U(C_t) = C_t^{1-\sigma^{-1}}/(1-\sigma^{-1})$. The last term of the RHS of (A.1) leads to

$$\begin{aligned} b_3 E_t \sum_{j=0} \phi_1^j (q_{t+j,t+j+1} + \tilde{y}_{t+j+1} - \tilde{y}_{t+j}) &= b_3(1-\sigma^{-1}) E_t[(\tilde{y}_{t+1} - \tilde{y}_t) + \phi_1(\tilde{y}_{t+2} - \tilde{y}_{t+1}) \\ &\quad + \phi_1^2(\tilde{y}_{t+3} - \tilde{y}_{t+2}) + \ldots]. \end{aligned} \quad \text{(A.2)}$$

Recall that in obtaining the IS equation (1), we defined the natural rate of interest as $r_t^n \equiv \sigma^{-1}(E_t \tilde{y}_{t+1}^n - \tilde{y}_t^n)$, where $\tilde{y}_t^n$ stands for the natural rate of output. Using this definition, (A.2) can be rewritten as

$$\begin{aligned} & b_3(1-\sigma^{-1}) E_t[(y_{t+1} - y_t + \tilde{y}_{t+1}^n - \tilde{y}_t^n) + \phi_1(y_{t+2} - y_{t+1} + \tilde{y}_{t+2}^n - \tilde{y}_{t+1}^n) + \ldots] \\ = \; & b_3(1-\sigma^{-1})[(y_{t+1}^e - y_t + \sigma r_t^n) + \phi_1(y_{t+2}^e - y_{t+1}^e + \sigma E_t r_{t+1}^n) + \ldots] \\ = \; & b_3(1-\sigma^{-1})[y_{t+1}^e - y_t + \sum_{j=2} \phi_1^{j-1}(y_{t+j}^e - y_{t+j-1}^e) + \sigma \frac{r_t^n}{1-\phi_1\rho_r}], \end{aligned}$$

where y_t denotes the output gap. Using the relation that $E_t \sum_{j=2} \phi_1^{j-1} y_{t+j-1}$ $= \phi_1 y_{t+1}^e + \phi_1 \sum_{j=2} \phi_1^{j-1} y_{t+j}^e$, the above equation leads to

$$b_3(1-\sigma^{-1})[(1-\phi_1)y_{t+1}^e - y_t + (1-\phi_1)\sum_{j=2}\phi_1^{j-1} y_{t+j}^e + \sigma \frac{r_t^n}{1-\phi_1\rho_r}].$$

It follows that

$$\begin{aligned} \pi_t &= [\zeta\tilde{\omega} - b_3(1-\sigma^{-1})]y_t + \tilde{b}_1 \pi_{t+1}^e + \tilde{b}_2 \phi_1 \sum_{j=2} \phi_1^{j-1} \pi_{t+j}^e + (1-\sigma^{-1}) b_3 (1-\phi_1) y_{t+1}^e \\ &+ (1-\sigma^{-1}) b_3 (1-\phi_1) \sum_{j=2} \phi_1^{j-1} y_{t+j}^e + (\sigma - 1) b_3 \frac{r_t^n}{1-\phi_1\rho_r}. \end{aligned} \quad \text{(A.3)}$$

[1] $\tilde{b}_1 = \phi_2[1 + \theta\chi(1+\omega)] - \phi_1\chi(\theta-1)$, $\tilde{b}_2 = \chi(\phi_2 - \phi_1)(\theta - 1)/\phi_1$ and $b_3 = \chi(\phi_2 - \phi_1)$.

where we used the relation $mc_t = \tilde{\omega} y_t$. This generalized NKPC and the IS equation can be summarized in a way consistent with eq.(7):

$$QX_t = WX_{t+1}^e + Ni_t + Ur_t^n + M\sum_{j=2}^{\infty}\phi_1^{j-1}X_{t+j}^e, \tag{A.4}$$

where

$$\begin{aligned} Q &= \begin{bmatrix} 1 & 0 \\ -\kappa + b_3(1-\sigma^{-1}) & 1 \end{bmatrix}, W = \begin{bmatrix} 1 & \sigma \\ b_3(1-\sigma^{-1})(1-\phi_1) & \tilde{b}_1 \end{bmatrix}, N = \begin{bmatrix} -\sigma \\ 0 \end{bmatrix}, \\ U &= \begin{bmatrix} \sigma \\ \frac{b_3(\sigma-1)}{1-\phi_1\rho_r} \end{bmatrix}, M = \begin{bmatrix} 0 & 0 \\ b_3(1-\sigma^{-1})(1-\phi_1) & \tilde{b}_2\phi_1 \end{bmatrix}, \end{aligned}$$

where $\kappa = \zeta\tilde{\omega}$.
Therefore, we can reexamine all the E-stability analyses done in the main text just by redefining the coefficient matrices. It should be noted that this generalized version is exactly the same as eq.(7) if we assume $b_3 = 0$ or $\sigma = 1$.
Under our benchmark parameter values, b_3 = -.001, .002, and .004 when annual trend inflation is -1%, 2%, and 4%, respectively. Figures 4 - 7, shown at the end of this appendix, suggest that introducing an additional term in the NKPC does not change the result because those figures are virtually identical to the corresponding ones shown in the main text.

B The contemporaneous expectations rule

Suppose that the policy rule is given by

$$i_t = F_{p\pi}\pi_t^e + F_{py}y_t^e, \tag{B.1}$$

where expectations are formed at $t-1$.
Following Bullard and Mitra (2002), the PLM is assumed to be given as

$$X_t = A_{t-1} + D_{t-1}r_{t-1}^n. \tag{B.2}$$

Notice that this is a simpler version of the PLM under the lagged-data rule ($C_{t-1} = 0$). It follows that

$$\sum_{j=2}\phi_1^{j-1}X_{t+j}^e = (1-\phi_1)^{-1}\phi_1 A_{t-1} + (1-\phi_1\rho_r)^{-1}\phi_1\rho_r^2 D_{t-1}r_{t-1}^n.$$

By defining $F_p = [F_{py}, F_{p\pi}]$, the ALM leads to

$$\begin{aligned} QX_t &= W(A_{t-1} + D_{t-1}\rho_r r_{t-1}^n) + NF_p(A_{t-1} + D_{t-1}r_{t-1}^n) \\ &+ Ur_t^n + M[(1-\phi_1)^{-1}\phi_1 A_{t-1} + (1-\phi_1\rho_r)^{-1}\phi_1\rho_r^2 D_{t-1}r_{t-1}^n]. \end{aligned} \tag{B.3}$$

It follows that

$$X_t = Q^{-1}\{(W + NF_p + (1-\phi_1)^{-1}\phi_1 M)A_{t-1} + [W\rho_r D_{t-1} + NF_p D_{t-1} + \rho_r U + (1-\phi_1\rho_r)^{-1}\phi_1 M\rho_r^2 D_{t-1}]r_{t-1}^n + U\varepsilon_t\}. \quad \text{(B.4)}$$

The T-maps are given by

$$T(A_{t-1}) = Q^{-1}(W + NF_p + (1-\phi_1)^{-1}\phi_1 M)A_{t-1} \quad \text{(B.5)}$$

$$T(D_{t-1}) = Q^{-1}\{[W\rho_r + NF_p + \phi_1(1-\phi_1\rho_r)^{-1}\rho_r^2 M]D_{t-1} + U\rho_r\}. \quad \text{(B.6)}$$

The E-stability condition is that all of the eigenvalues of $T(A)$ and $T(D)$ have real parts less than one. I show the analytical expression of the E-stability condition in the next section.

C Analytical expression of the E-stability conditions

C.1 The contemporaneous rule

From eqs. (9) and (10) of the main text, the rule based on contemporaneous data will be E-stable if and only if all of the eigenvalues of the following two matrices have real parts less than one:

$$DT_A = \Delta \begin{bmatrix} 1 & \sigma - \sigma F_{c\pi}[b_1 + \frac{\phi_1}{1-\phi_1}b_2] \\ \kappa & \kappa\sigma + (1+\sigma F_{cy})[b_1 + \frac{\phi_1}{1-\phi_1}b_2] \end{bmatrix}$$

and

$$DT_D = \rho_r\Delta \begin{bmatrix} 1 & \sigma - \sigma F_{c\pi}[b_1 + \frac{\phi_1\rho_r}{1-\phi_1\rho_r}b_2] \\ \kappa & \kappa\sigma + (1+\sigma F_{cy})[b_1 + \frac{\phi_1\rho_r}{1-\phi_1\rho_r}b_2] \end{bmatrix},$$

where $\Delta \equiv 1/(1+\sigma F_{cy} + \sigma\kappa F_{c\pi})$. Suppose that the characteristic equation of $DT_A - I$ is given as $\mu^2 + m_1\mu + m_2 = 0$, where $m_1 = -tr(DT_A - I)$ and $m_2 = det(DT_A - I)$. As is pointed out by Bullard and Mitra (2002), both of the eigenvalues of DT_A have real parts less than one if and only if $m_1 > 0$ and $m_2 > 0$. It follows that

$$1 + \sigma(2F_{cy} + 2\kappa F_{c\pi} - \kappa) - (1+\sigma F_{cy})(b_1 + \frac{\phi_1}{1-\phi_1}b_2) > 0, \quad \text{(C.1)}$$

$$\kappa(F_{c\pi} - 1) + F_{cy}(1 - b_1 - \frac{\phi_1}{1-\phi_1}b_2) > 0. \quad \text{(C.2)}$$

The corresponding conditions for DT_D are

$$1 + \sigma(2F_{cy} + 2\kappa F_{c\pi} - \kappa) - (1+\sigma F_{cy})(b_1 + \frac{\rho_r\phi_1}{1-\rho_r\phi_1}b_2) > 0, \quad \text{(C.3)}$$

$$\kappa(F_{c\pi} - 1) + F_{cy}(1 - b_1 - \frac{\rho_r\phi_1}{1-\rho_r\phi_1}b_2) > 0. \quad \text{(C.4)}$$

We obtain the following proposition:

Proposition C.1 *Assume that $\phi_1 = \alpha\beta\bar{\Pi}^{\theta-1} < 1$ and $\rho_r < 1$. The E-stability condition for the policy rule under the contemporaneous data is given by (C.1) and (C.2) if $b_2 \geq 0$, and (C.3) and (C.4) otherwise.*

Thus, the E-stability condition under nonzero trend inflation is generally stipulated by four inequalities.[2] Notice that in the special case of zero trend inflation, where $b_1 = \beta$ and $b_2 = 0$, the E-stability condition reduces to $\kappa(F_{c\pi} - 1) + F_{cy}(1 - \beta) > 0$. In the generalized version, the term $(1 - \beta)$ is replaced with $1 - b_1 - \phi_1 b_2/(1 - \phi_1)$, which describes the slope of the NKPC under nonzero trend inflation.

C.2 The forecast-based rule

DT_A and DT_D under the forecast-based rule are given as

$$DT_A = \begin{bmatrix} 1 - \sigma F_{fy} & \sigma(1 - F_{f\pi}) \\ \kappa(1 - \sigma F_{fy}) & \kappa\sigma(1 - F_{f\pi}) + b_1 + \frac{\phi_1}{1-\phi_1} b_2 \end{bmatrix}$$

and

$$DT_D = \rho_r \begin{bmatrix} 1 - \sigma F_{fy} & \sigma(1 - F_{f\pi}) \\ \kappa(1 - \sigma F_{fy}) & \kappa\sigma(1 - F_{f\pi}) + b_1 + \frac{\rho_r\phi_1}{1-\rho_r\phi_1} b_2 \end{bmatrix}.$$

Conditions $-tr(DT_A - I) > 0$ and $\det(DT_A - I) > 0$ lead to

$$\sigma F_{fy} + \kappa\sigma(F_{f\pi} - 1) + 1 - b_1 - \frac{\phi_1}{1 - \phi_1} b_2 > 0, \tag{C.5}$$

$$\kappa(F_{f\pi} - 1) + F_{fy}(1 - b_1 - \frac{\phi_1}{1 - \phi_1} b_2) > 0. \tag{C.6}$$

The corresponding conditions for DT_D are

$$\sigma F_{fy} + \kappa\sigma(F_{f\pi} - 1) + 1 - b_1 - \frac{\rho_r\phi_1}{1 - \rho_r\phi_1} b_2 > 0, \tag{C.7}$$

$$\kappa(F_{f\pi} - 1) + F_{fy}(1 - b_1 - \frac{\rho_r\phi_1}{1 - \rho_r\phi_1} b_2) > 0. \tag{C.8}$$

If b_2 is non-negative (negative), then the only relevant conditions are (C.5) and (C.6) ((C.7) and (C.8)). This suggests that the E-stability condition under the forecast-based rule is generally different from the one under the contemporaneous rule, whereas the forecast-based rule and the contemporaneous rule share the E-stability conditions in common under zero trend inflation.

Proposition C.2 *Assume that $\phi_1 = \alpha\beta\bar{\Pi}^{\theta-1} < 1$ and $\rho_r < 1$. The E-stability condition for the policy rule under the forecast-based data is given by (C.5) and (C.6) if $b_2 \geq 0$, and (C.7) and (C.8) otherwise.*

[2] It can be shown that b_2 may take a negative value when $\bar{\Pi}$ is sufficiently large even if $\phi_1 < 1$.

C.3 The contemporaneous expectations rule

Under the contemporaneous expectations rule, DT_A and DT_D are given as

$$DT_A = \begin{bmatrix} 1-\sigma F_{py} & \sigma(1-F_{p\pi}) \\ \kappa(1-\sigma F_{py}) & \kappa\sigma(1-F_{p\pi}) + b_1 + \frac{\phi_1}{1-\phi_1}b_2 \end{bmatrix}$$

and

$$DT_D = \rho_r \begin{bmatrix} 1-(\sigma/\rho_r)F_{py} & \sigma(1-F_{p\pi}/\rho_r) \\ \kappa(1-(\sigma/\rho_r)F_{py}) & \kappa\sigma(1-F_{p\pi}/\rho_r) + b_1 + \frac{\rho_r\phi_1}{1-\rho_r\phi_1}b_2 \end{bmatrix}.$$

It is clear that the combination of policy coefficients that satisfy $-tr(DT_A - I) > 0$ and $\det(DT_A - I) > 0$ are exactly the same as those under the forecast-based rule:

$$\sigma F_{py} + \kappa\sigma(F_{p\pi} - 1) + 1 - b_1 - \frac{\phi_1}{1-\phi_1}b_2 > 0, \tag{C.9}$$

$$\kappa(F_{p\pi} - 1) + F_{py}(1 - b_1 - \frac{\phi_1}{1-\phi_1}b_2) > 0. \tag{C.10}$$

On the other hand, the conditions $-tr(DT_D - I) > 0$ and $\det(DT_D - I) > 0$ yield

$$\frac{\sigma}{\rho_r}F_{py} + \kappa\sigma\left(\frac{1}{\rho_r}F_{p\pi} - 1\right) + 1 - b_1 - \frac{\rho_r\phi_1}{1-\rho_r\phi_1}b_2 > 0, \tag{C.11}$$

$$\kappa\left(\frac{1}{\rho_r}F_{p\pi} - 1\right) + \frac{1}{\rho_r}F_{py}(1 - b_1 - \frac{\rho_r\phi_1}{1-\rho_r\phi_1}b_2) > 0. \tag{C.12}$$

If $b_2 \geq 0$, then condition (C.11) is redundant because the inequality always holds as long as (C.9) is satisfied. The difference between the LHSs of (C.10) and (C.12) ((C.10) - (C.12)) leads to

$$\left(1 - \frac{1}{\rho_r}\right)(\kappa F_{p\pi} + (1-b_1)F_{py}) - \frac{\phi_1^2(1-\rho_r)}{(1-\phi_1)(1-\rho_r\phi_1)}b_2 F_{py} < 0 \quad \text{if} \quad b_1 \leq 1, b_2 \geq 0, \kappa \geq 0.$$

It follows that the E-stability condition is given by (C.9) and (C.10) if $b_1 \leq 1$, $b_2 \geq 0$ and $\kappa \geq 0$. This suggests that the E-stability condition under the contemporaneous expectations rule becomes exactly the same as that under the forecast-based rule. More generally, however, any of the conditions (C.9) - (C.12) can be relevant.

Proposition C.3 *Assume that $\phi_1 = \alpha\beta\bar{\Pi}^{\theta-1} < 1$ and $\rho_r < 1$. The E-stability condition for the policy rule based on contemporaneous expectations is given by (C.9) and (C.10) if $b_1 \leq 1$, $b_2 \geq 0$ and $\kappa \geq 0$, and (C.9) - (C.12) otherwise.*

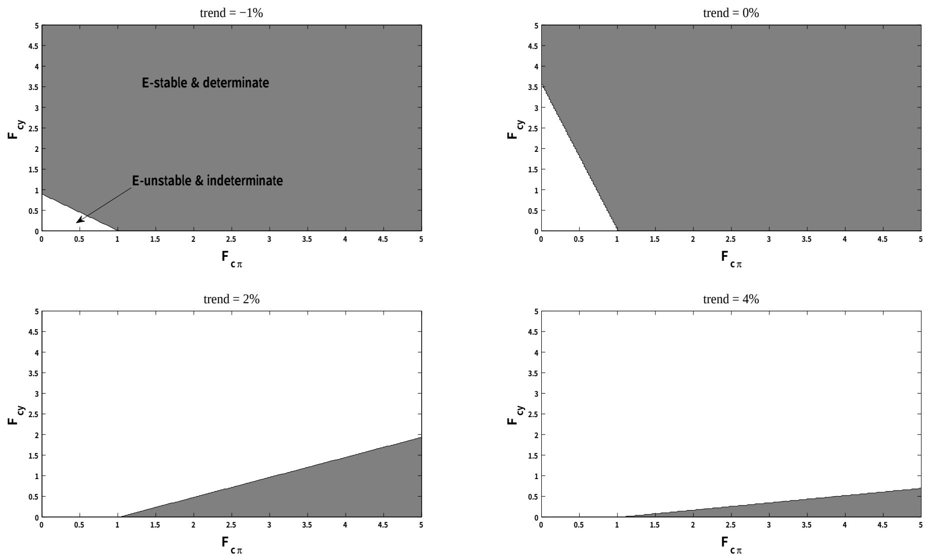

Figure 4: The E-stability and determinacy regions under the contemporaneous rule: $b_3 > 0$.

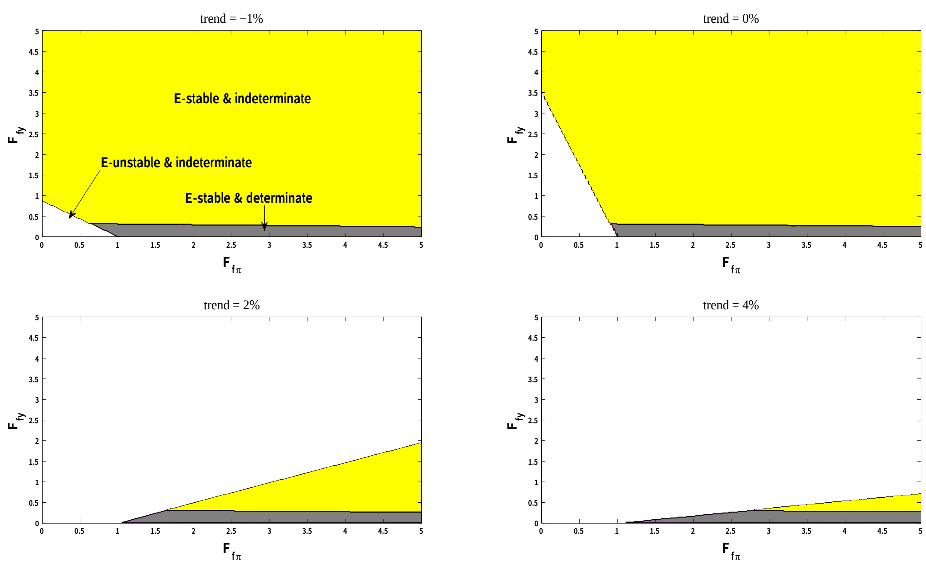

Figure 5: The E-stability and determinacy regions under the forecast-based rule: $b_3 > 0$.

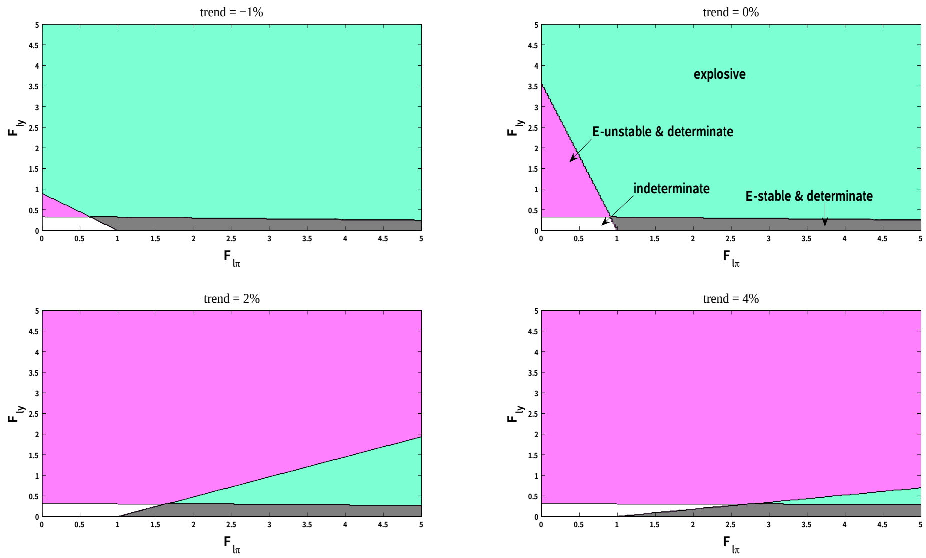

Figure 6: The E-stability and determinacy regions under the lagged-data rule: $b_3 > 0$.

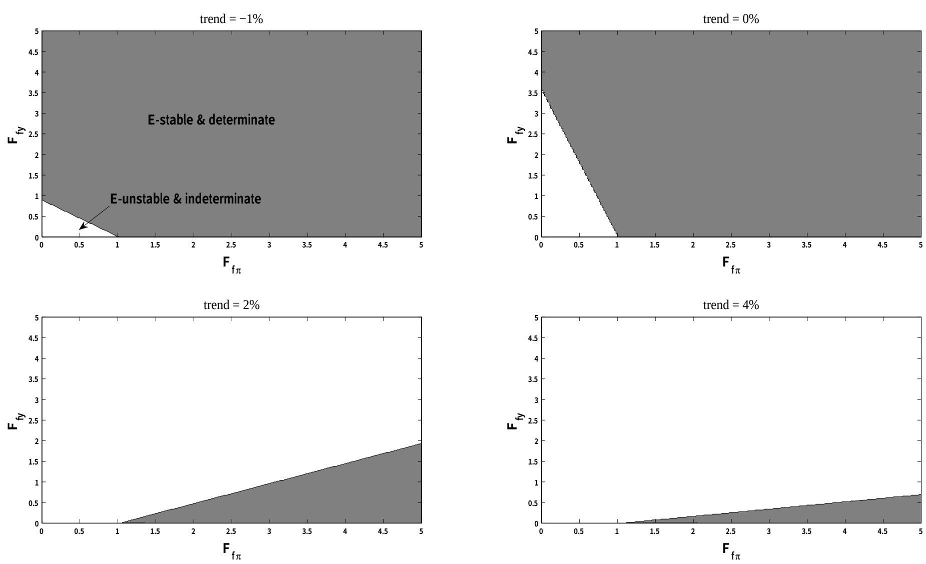

Figure 7: The E-stability and determinacy regions under the contemporaneous expectations rule: $b_3 > 0$.